M.R P. 1040
S.C. 11/230
19th February, 1957

METEOROLOGICAL OFFICE
17 JUN 1957
LERWICK OBSERVATORY
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The representation of circumpolar 500 mb charts by a series of spherical harmonics

by

A. Gilchrist

(Meteorological Office)

AIR MINISTRY

METEOROLOGICAL RESEARCH COMMITTEE

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1. Introduction

Though it was intended to use the results to investigate the feasibility of making a numerical index of charts, the mathematical representation of 500 mb charts was undertaken mainly with a view to using the representations in numerical forecasting. It is hoped that this will reduce the number of variables that have to be dealt with (as compared with finite difference methods of integrating the dynamical equations), and thus lead to a method applicable to medium range forecasting. The representations have been made in terms of surface spherical harmonics, because, being functions of co-latitude and longitude only on a sphere, they are naturally suited to the purpose, and because both the derivation of the geostrophic vorticity from the height field and the reverse process are then comparatively simple. Previous work on the use of spherical harmonics to represent meteorological charts has been reported by Haurwitz and Craig (1952) and Blinova (1943).

2. Surface spherical harmonics

Surface spherical harmonics Y_n^m of integral degree n, and order m satisfy the equation,

$$\nabla_s^2 Y_n^m + \frac{n(n+1)}{R^2} Y_n^m = 0 \qquad \text{---- (1)}$$

where ∇_s^2 is the Laplacian operator on the surface of the sphere of radius R, and we take m to be a positive integer less than or equal to n (a positive integer), or zero.

The usual form taken for the Y_n^m's is

$$P_n^m(\cos\theta)\left\{ a_n^m \cos m\phi + b_n^m \sin m\phi \right\}$$

where θ is the co-latitude, ϕ is the longitude and a_n^m and b_n^m are arbitrary constants. $P_n^m(\cos\theta)$ is the Associated Legendre Function of degree n and order m. We shall deal throughout with the semi-normalised functions which are such that the mean value of $\left\{ P_n^m(\cos\theta) \begin{matrix} \sin m\phi \\ \cos m\phi \end{matrix} \right\}^2$ over the unit sphere is $\frac{1}{2n+1}$. These functions have been tabulated to P_6^6 by Schmidt (1935). For the present investigation the tables were extended to P_{12}^{12} using the coefficients calculated by Doodson (1936). They were evaluated on the Manchester University Electronic Computer.

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It may be helpful to recall briefly the form of the function

$$P_n^m(\cos\theta)\left\{a_n^m \cos m\varphi + b_n^m \sin m\varphi\right\}$$

It vanishes along $(n-m)$ circles of latitude, symmetrically placed with respect to the equator, and if $m \neq 0$, its value at the poles is also zero; also along m great circles through the poles, the angle between any two consecutive great circles being π/m. If $m = 0$, the function is called a zonal harmonic, and the value along any line of latitude is constant. If $m = n$, there are no zeros along lines of latitude, and the function is called a sectorial harmonic. Otherwise the function is called a tesseral harmonic. For illustration Fig.1 shows $P_8^2 \sin 2\phi$ drawn on a circumpolar chart.

Spherical harmonics are orthogonal and have the property that in fitting a completely defined function, the coefficient of a harmonic found by least squares does not depend on the number of harmonics used to fit the function. This is true for instance also in ordinary Fourier analysis where if $f(\theta)$ is perfectly represented by the infinite series

$$a_0 + \sum_{r=1}^{\infty}\left(a_r \cos r\theta + b_r \sin r\theta\right)$$

and is approximated by a series containing $(2s + 1)$ coefficients which are determined by least squares so that

$$f(\theta) \doteq a'_0 + \sum_{r=1}^{s}\left(a_r \cos r\theta + b_r \sin r\theta\right)$$

then $$a_0 = a'_0 \quad , \quad a_r = a'_r \quad , \quad b_r = b'_r$$

For ordinary Fourier analysis this is also true when the initial function is not known everywhere, but only at equally spaced values of θ. However, it is not true in this case for spherical harmonics; if a function is known at a network of points over a sphere the values of the coefficients found by the least squares criterion are dependent on the number of harmonics.

The method of spherical harmonic analysis used in the cases so far considered is that due to Schuster (1902). It involves the following steps

(1) plain harmonic analysis of the contour heights around lines of latitude, giving coefficients $a_m(\theta_l)$, $b_m(\theta_l)$ where θ_l denotes the particular line of latitude, and m the order of the harmonic

(2) a modified harmonic analysis of corresponding harmonic coefficients $a_m(\theta_1), a_m(\theta_2)$ ... or $b_m(\theta_1), b_m(\theta_2)$ ... given by (1); if m is even the analysis being carried out in sines only, if odd in cosines only

(3) the use of Schuster's coefficients, which effectively substitute for $\cos r\theta$ or $\sin r\theta$ in terms of Associated Legendre Polynomials of any given order. Schuster gives the coefficients necessary to determine the coefficients up to P_{12}^{12}. His method is not a least squares method. The coefficients obtained by it do not depend on the number of harmonics used in the representation and are, in general, not the best possible ones. However it was considered that in the first stages of this investigation, flexibility and ease of fitting were more important than best possible fit, and Schuster's method has just these advantages.

The harmonic coefficients a_n^m, b_n^m given by Schuster's method depend only on the values of the $a_m(\theta)$ and $b_m(\theta)$ of stage (1).

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That is, $\{A_m(\theta)\cos m\phi + B_m(\theta)\sin m\phi\}$ is fitted by

$$\left\{a_m^m P_m^m + a_{m+2}^m P_{m+2}^m + \ldots\right\}\cos m\phi + \left\{b_m^m P_m^m + b_{m+2}^m P_{m+2}^m + \ldots\right\}\sin m\phi$$

where $\sum_{m=0}^{\infty}\{A_m(\theta)\cos m\phi + B_m(\theta)\sin m\phi\}$ defines the initial function completely. For this reason, when considering the goodness of fit, it is convenient to treat each order m separately.

3. The representations

Four 500 mb hemispherical charts have been analysed, viz., those for

I	0300Z	28th January, 1953	II	0300Z	9th November, 1953
III	0300Z	27th February, 1953	IV	0300Z	29th June, 1953.

These data were analysed because they had previously been subjected to plane harmonic analysis along lines of latitude. Initially the values of the 500 mb heights were read off at 5° intervals of latitude and longitude from circumpolar charts. Near the equator, where no values were given on the charts, convenient values were supplied. It was assumed that the Southern Hemisphere pattern was a mirror image of the Northern Hemisphere one, and so the representations involve only even spherical harmonics (i.e., $n-m$ is even). The fitting of the four cases was carried out on the Manchester University Electronic Computer; a programme which synthesised the charts, given the spherical harmonic coefficients, was also operated.

Fig. 2, 5, 8, 11 show the circumpolar charts which were represented; Fig. 3, 6, 9, 12 the representations with harmonics up to P_{12}^{12} (91 coefficients); Fig. 4, 7, 10, 13 the representations with harmonics up to P_8^8 (45 coefficients). The fitting is considerably more successful in cases I and III than in II and especially IV. Case IV is a summer situation and in summer the average wavelength between troughs is less than in other seasons. Generally, short wavelength features are absent from or only inadequately represented on the reconstructed charts. To fit them adequately would require the inclusion of higher harmonics, though a better method of fitting (e.g., a least squares method with some weighting towards middle latitudes) should be able to include more of them without an increase in the number of coefficients.

It is usual in measuring the adequacy of a fit by ordinary harmonic analysis to make use of the theorem that, if x_i are $(2n+1)$ equally spaced values, $\bar{x}$ is their mean, and A_i, B_i are the harmonic coefficients of order i derived by least squares then

$$\sum_{i=0}^{2n}(x_i-\bar{x})^2 = \sum_{i=1}^{n}(A_i^2+B_i^2)$$

There is an equivalent integral theorem for surface spherical harmonics; this is

$$\int_S f^2\,ds = (a_0^0)^2 + \sum_{n=1}^{\infty}\sum_{m=0}^{n}\left\{(a_n^m)^2+(b_n^m)^2\right\}$$

where the integral of f^2 is taken over the surface of the sphere and a_p^q, b_p^q are the coefficients of the normalised spherical harmonics of order q and degree p fitted by least squares.

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The corresponding equation for discrete values of the function f is not strictly true, and, further, does not hold in the present examples because Schuster's method is not a least squares one. However it has been assumed to hold sufficiently nearly to provide a reasonable method of judging goodness of fit when used comparatively; for instance, from it we can obtain information about whether one order is fitted more closely than another; but, on the other hand, it would be wrong to assume that if, for example, the sum of the squares of the normalised harmonic coefficients came to 99 per cent of the initial total variance, then only 1 per cent remained to be fitted.

Tables I-IV summarise the variances and residual variances for the four cases.

Table I - The variance of the initial function over the sphere omitting that due to the variation of mean height in the N.-S. direction

	Case I	Case II	Case III	Case IV	
V_T (Total variance)	117,347	96,884	129,844	38,968	
V_{12} (Variance up to harmonic 12)	116,583	95,928	128,702	38,626	
$V_{12}/V_T \times 100$	99.35%	99.01%	99.12%	99.12%	Units $(ft)^2$
V_8 (Variance up to harmonic 8)	114,984	94,802	127,394	36,961	
$V_2/V_T \times 100$	97.99%	97.85%	98.11%	94.85%	

These variances have been derived from the plane harmonic analysis along lines of latitude.

$$V_T = \frac{\sum_\theta V_\theta \sin\theta}{\sum_\theta \sin\theta} = \frac{\sum_\theta \sum_n \left\{ a_n^2(\theta) + b_n^2(\theta) \right\} \sin\theta}{\sum_\theta \sin\theta}$$

where θ is the co-latitude and V_θ is the total variance along the line of latitude defined by θ = const. V_{12} and V_8 are derived in the same way, but using the variance accounted for by harmonics 0-12 and 0-8 respectively, in the initial plane harmonic analysis.

Table II - Variance due to the N.-S. variation of mean 500 mb height averaged around a line of latitude

	Case I	Case II	Case III	Case IV	
V_T (Total variance)	1170510	1066409	1536725	174411	
V_{12}	1129	1232	452	49	Units $(ft)^2$
V_8	3399	3129	132	255	

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These variances were derived giving equal weight to each line of latitude

$$\text{i.e.,}\quad V_T = \frac{1}{n}\sum_{\theta}\left\{\bar{h}_\theta - \frac{\sum_{\theta}\bar{h}_\theta}{n}\right\}^2$$

where n is the number of lines of latitude considered and $\bar{h}_\theta$ is the mean height around the line of latitude θ = constant. V_{12} and V_8 are the variances not accounted for by the zonal harmonics 0-12 and 0-8 respectively,

$$\text{i.e.,}\quad V_{12} = \frac{1}{n}\sum\left(\bar{h}_\theta - \sum_{0}^{12} a_n P_n^0\right)^2$$

/Table III

Table III - The variance associated with the separate orders in the plane harmonic analysis, and the amount not accounted for in the spherical harmonic analysis (neglecting N.-S. variation of mean 500 mb height)

		Case I		Case II		Case III		Case IV		
		variance	% of total	variance	% of total	variance	% of total	variance	% of total	
Order 1	V_{T1} x 100	54621	46.55	42192	43.55	22246	17.13	92244	23.67	
	V_{12}	555	.46	612	.63	142	.11	318	1.59	
	V_8	2669	2.27	3597	3.71	3758	2.89	1658	4.25	
2	V_{T2}	19969	17.02	19350	19.97	61097	47.05	7634	19.59	
	V_{12} x 100	1079	.92	1227	1.27	179	.14	1280	3.28	
	V_8	1206	1.03	1447	1.49	356	.27	1520	3.90	
3		17868	15.23	6502	6.71	32313	24.89	7359	18.88	
		858	.73	1307	1.35	510	.39	372	.97	
		3031	2.58	2583	2.67	1294	1.00	1308	3.35	
4		12211	10.41	5563	5.74	1810	1.39	8211	21.07	
		394	.34	14	.01	430	.33	227	.58	
		2536	2.16	3556	3.67	910	.70	1923	4.93	
5		3175	2.71	9264	9.56	4361	3.36	1610	4.13	
		279	.24	1344	1.39	382	.29	840	2.18	
		2994	2.55	6084	6.28	2059	1.59	1444	3.71	
6		3352	2.86	1365	1.41	3503	2.70	968	2.48	
		208	.18	163	.17	826	.64	207	.53	
		1332	1.14	507	.52	1998	1.54	406	1.04	
7		2788	2.38	9365	9.67	1209	.93	1536	3.94	Units $(ft)^2$
		1185	1.01	1755	1.81	902	.69	248	.64	
		2671	2.28	8615	8.89	1124	.87	1485	3.81	
8		1021	.87	1053	1.09	851	.66	399	1.02	
		592	.50	678	.70	714	.55	239	.61	
		882	.75	1035	1.07	849	.65	381	.98	
9		520	.44	504	.52	745	.57	1196	3.07	
		134	.11	421	.43	471	.36	803	2.06	
		520	.44	504	.52	745	.57	1196	3.07	
10		513	.44	488	.50	312	.24	216	.55	
		460	.39	425	.44	201	.15	179	.46	
		513	.44	488	.50	312	.24	216	.55	
11		349	.30	181	.19	167	.13	65	.17	
		342	.29	178	.18	165	.13	63	.16	
		349	.30	181	.19	167	.13	65	.17	
12		222	.19	168	.17	80	.06	189	.48	
		215	.18	166	.17	70	.06	189	.48	
		222	.19	168	.17	80	.06	189	.48	
% of total variance accounted for using harmonics up to 12		92.7		90.4		95.3		86.3		

$$V_{Tm} = \frac{\sum_{\theta} V_{\theta m} \sin\theta}{\sum_{\theta} \sin\theta} = \frac{\sum_{\theta} \{a_m^2(\theta) + b_m^2(\theta)\} \sin\theta}{\sum_{\theta} \sin\theta}$$

where $V_{\theta m}$ is the variance around line of latitude θ = constant due to the m^{th} harmonic.

$$(V_{12})_m = V_{Tm} - \sum_{n=m}^{12} \frac{\{(a_n^m)^2 + (b_n^m)^2\}}{2n+1}$$

$$(V_8)_m = V_{Tm} - \sum_{n=m}^{8} \frac{\{(a_n^m)^2 + (b_n^m)^2\}}{2n+1}$$

Table IV shows the average and the maximum percentage (over the 4 cases) of the total variance still to be accounted for in the various orders after fitting with harmonics up to P_{12}^{12}.

Table IV - Percentage variance in each order not accounted for in the fitting

Order	1	2	3	4	5	6	7	8	9	10	11	12
Average	.70	1.40	.86	.32	1.0	.38	1.04	.59	.74	.36	.19	.22
Maximum	1.59	3.28	1.35	.58	2.18	.64	1.81	.70	2.06	.46	.29	.48
No. of harmonics	6	6	5	5	4	4	3	3	2	2	1	1

It is seen that the greatest average percentages occur for orders 2,3, 5 and 7. The increase between 6 and 7 is shown by all four cases, and presumably arises because of the decrease from 4 to 3 in the number of harmonics. For order 7 there are only harmonics P_7^7, P_9^7, P_{11}^7 and of these, as explained below, P_7^7 makes a negligible contribution. The maximum for order 9 is for case IV, the only summer situation represented. In summer a larger number of troughs is likely to appear in the hemispherical pattern than at other periods of the year, and this must lead to an increased prominence of the higher harmonics.

It is notable how small a part the sectorial harmonics play in the representations. This is not surprising in view of their form over the sphere. Table V shows the average and maximum contribution of the sectorial harmonics to the total variance.

Table V - Contribution of sectorial harmonics to the total variance

Order	1	2	3	4	5	6	7	8	9	10	11	12
Average	4.6%	1.8%	2.2	.3	.2	.2	.3	0	.1	0	0	0
Maximum	12.2	4.6	3.6	.6	.4	.3	.8	.1	.1	0	0	0

The contribution becomes negligible after the third or fourth harmonic, and we could clearly omit most of the sectorial harmonics without significant loss.

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In judging the adequacy of the representations, it has to be remembered that when using them to provide numerical forecasts, it is necessary to derive the vorticity field from the height field. The variance contribution of a particlar harmonic to the vorticity field is $\{n(n+1)\}^2$ times its contribution to the height field, where n is the degree of the harmonic, so that terms whose numerical values are small may be important in the vorticity field if they are associated with a large n. In practice, it might be expedient to use different numbers of harmonics for the height and vorticity fields in the forecasting problem. We would then attempt to include a small wavelength feature in the vorticity field, while omitting it from the wind field.

4. Use of spherical harmonic representations

The representations might be used to select circumpolar charts which have basic similarities, possibly for use in some form of analogue forecasting. The spherical harmonic coefficients a_n^m, b_n^m would then form the basis of a numerical index, and analogous charts would be found by considering the magnitudes of corresponding coefficients.

It is intended to use the representations to solve the barotropic equation for the hemisphere. The equation which will be dealt with is

$$\nabla^2 \frac{\partial h}{\partial t} = -\frac{f}{g} J(h, \nabla_s^2 h) - J(h, f)$$

when h is the contour height and f the Coriolis parameter.

If h has been expressed in terms of spherical harmonics, $\nabla_s^2 h$ is easily obtained from equation (1). The right hand side then contains products of spherical harmonics; with suitable 'interaction' coefficients it may be expressed as a sum of spherical harmonics. Then using the integral form of equation (1)

$$\frac{1}{\nabla_s^2} Y_n^m = \frac{R^2}{n(n+1)} Y_n^m + F, \quad \text{where } \nabla^2 F = 0$$

$\partial h/\partial t$ can be found. The success of the method depends upon obtaining a sufficiently accurate initial representation without the number of 'interaction' coefficients becoming prohibitively large.

5. Conclusions

By using the even spherical harmonics up to P_8^8 and P_{12}^{12}, adequate representation of the largest features of the 500 mb hemispherical flow have been obtained, but even in the representations using harmonics to P_{12}^{12}, some features which appear to be important synoptically are not reproduced in the representations.

Further investigations using more harmonics, and better methods of fitting are needed.

Acknowledgement

The author is indebted to Miss M. K. Hinds and Mr. F. H. Bushby for help in preparing and testing the electronic computer programmes.

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Fig. 1
DIAGRAM OF $P_8^2 \sin 2\Phi$

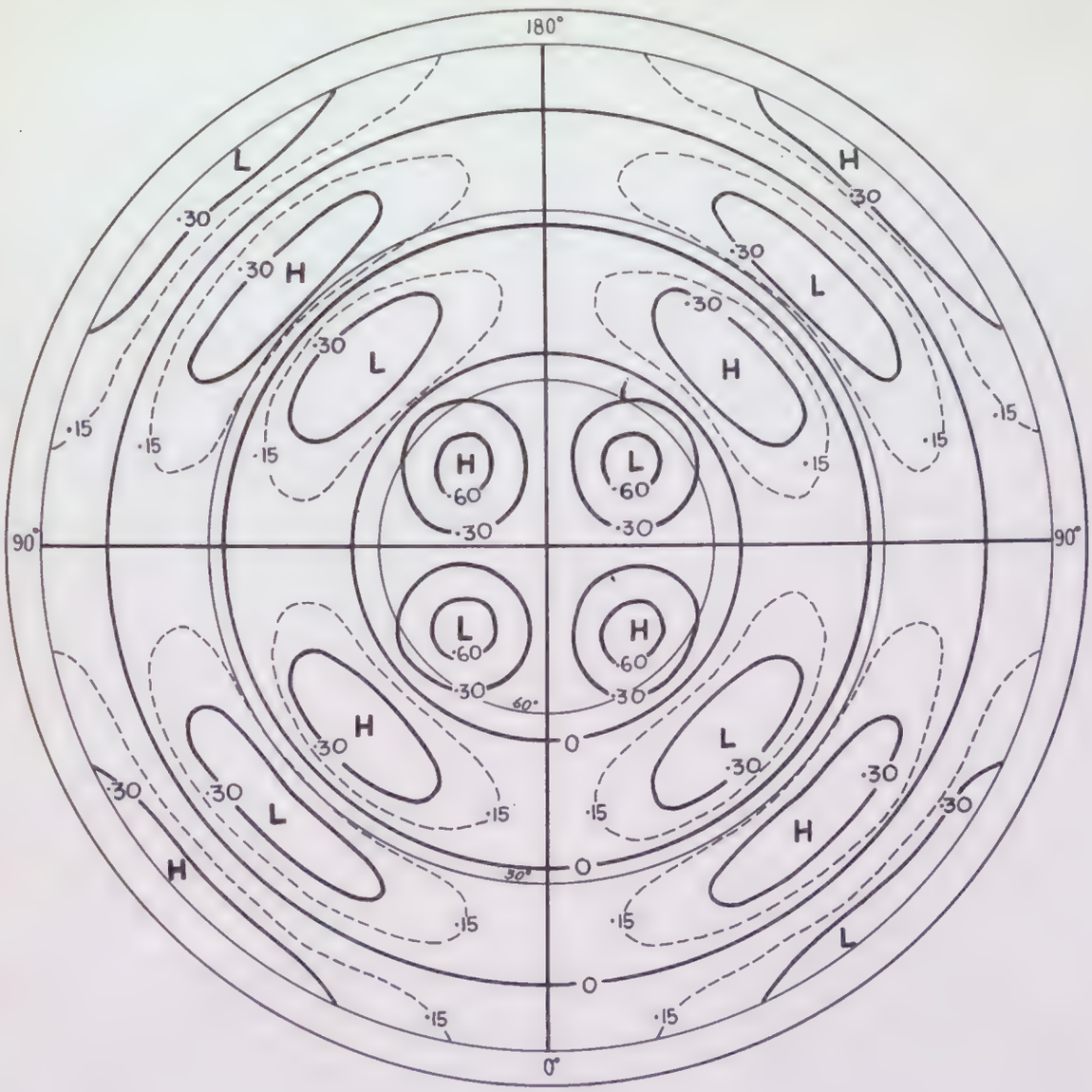


FIG. 2 500mb. 0300Z 28:1:53

FIG. 3 500mb. 0300Z 28:1:53

SPHERICAL HARMONIC REPRESENTATION $n,m,\leq 12$

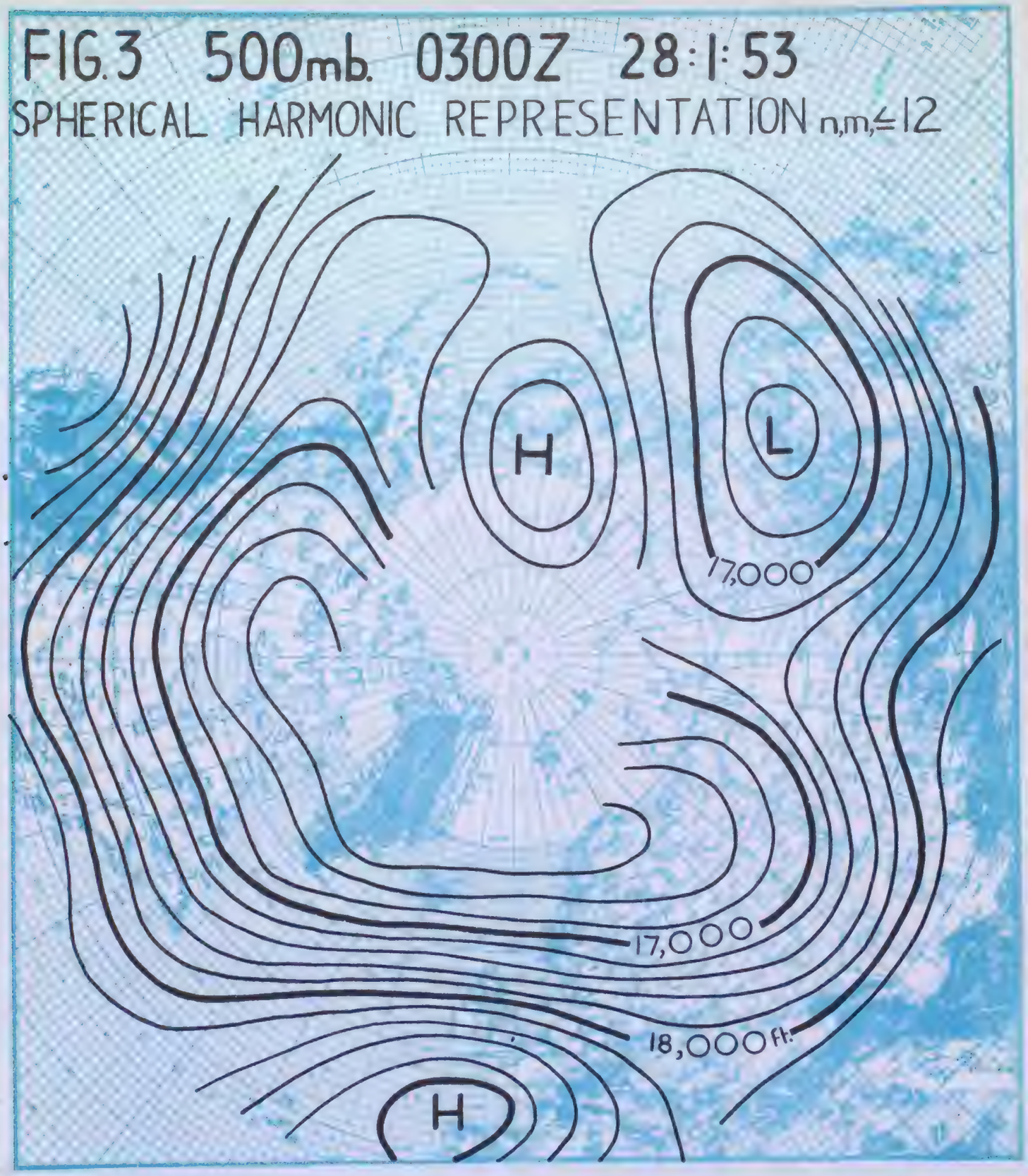


FIG.4 500mb. 0300Z 28:1:53
SPHERICAL HARMONIC REPRESENTATION $n,m \leq 8$

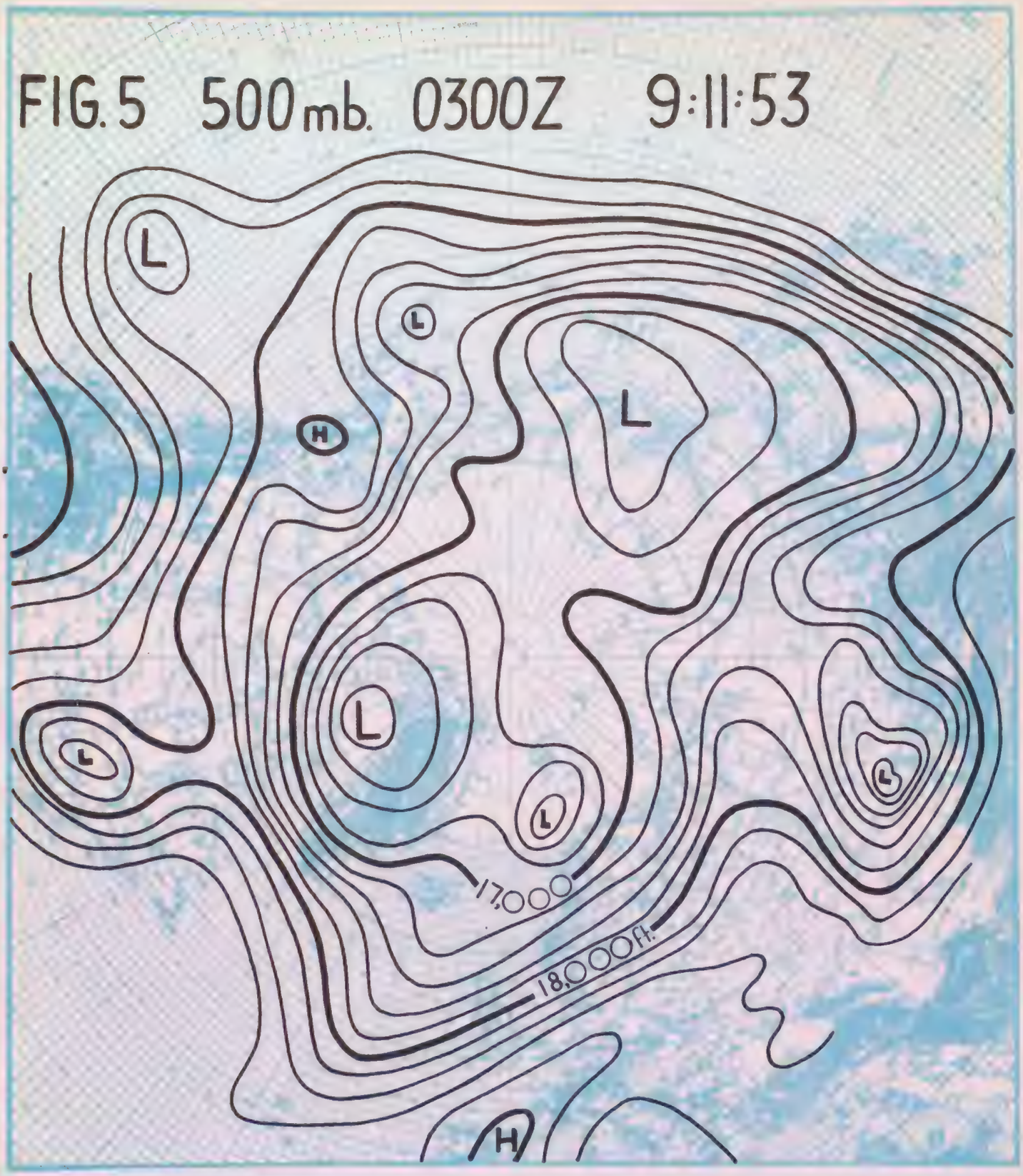


FIG. 5 500 mb. 0300Z 9:11:53
L
L
L
H
L
L
L
L
L
17,000
18,000 Ft.
H

FIG. 6 500mb. 0300Z 9:11:53

SPHERICAL HARMONIC REPRESENTATION $n,m \leq 12$

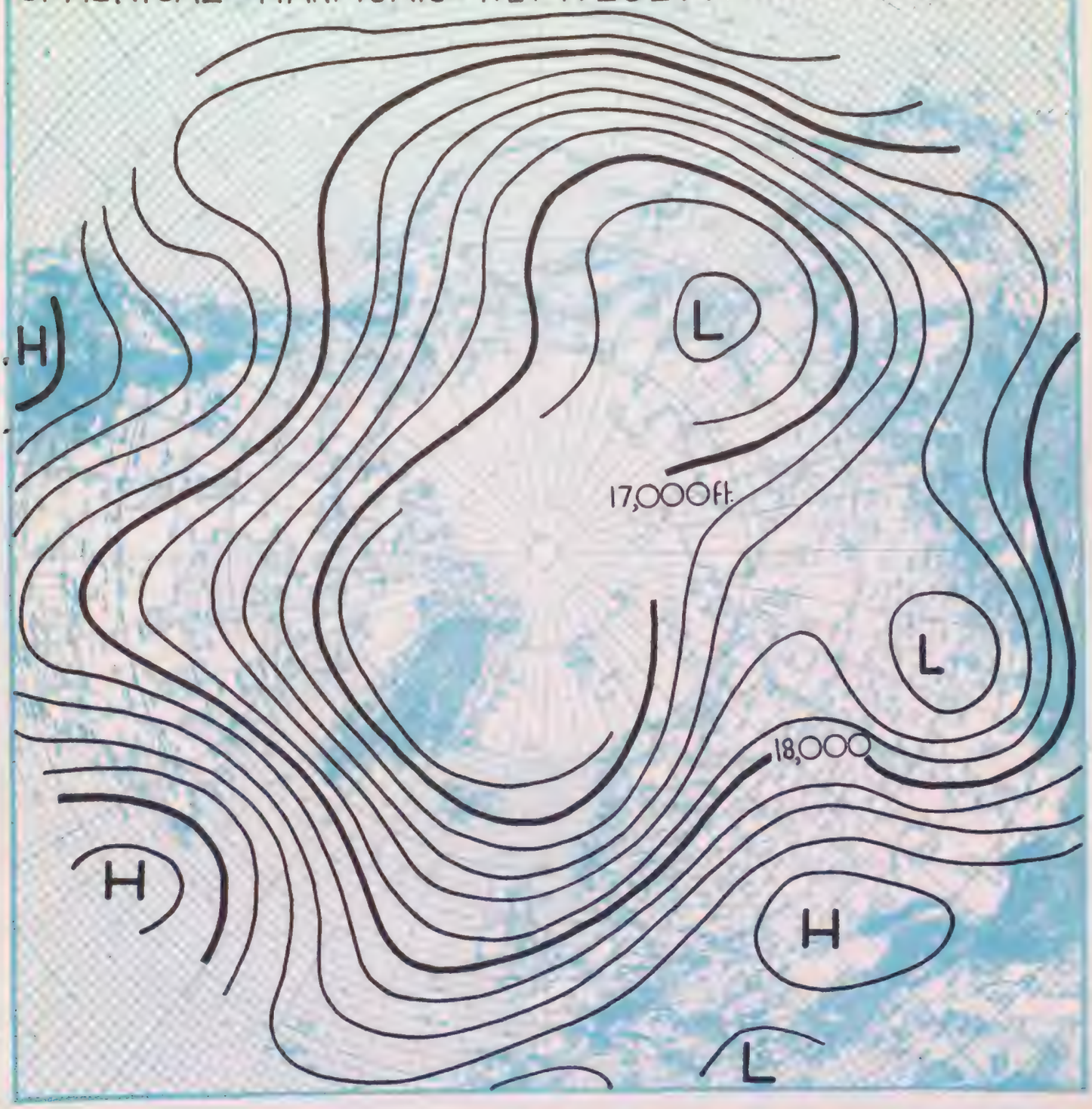
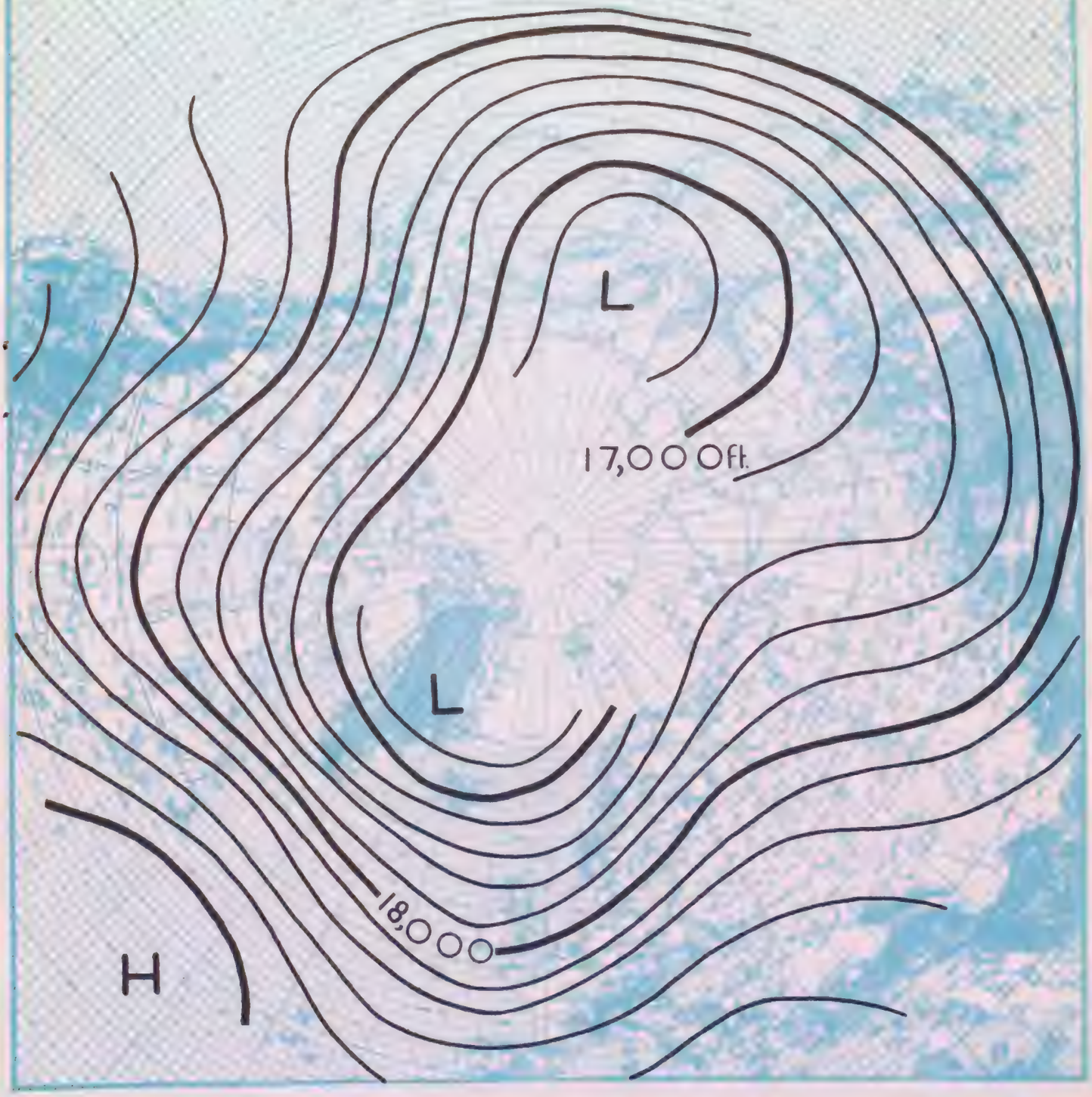


FIG.7 500mb. 0300Z 9:11:53

SPHERICAL HARMONIC REPRESENTATION $n,m,\leq 8$



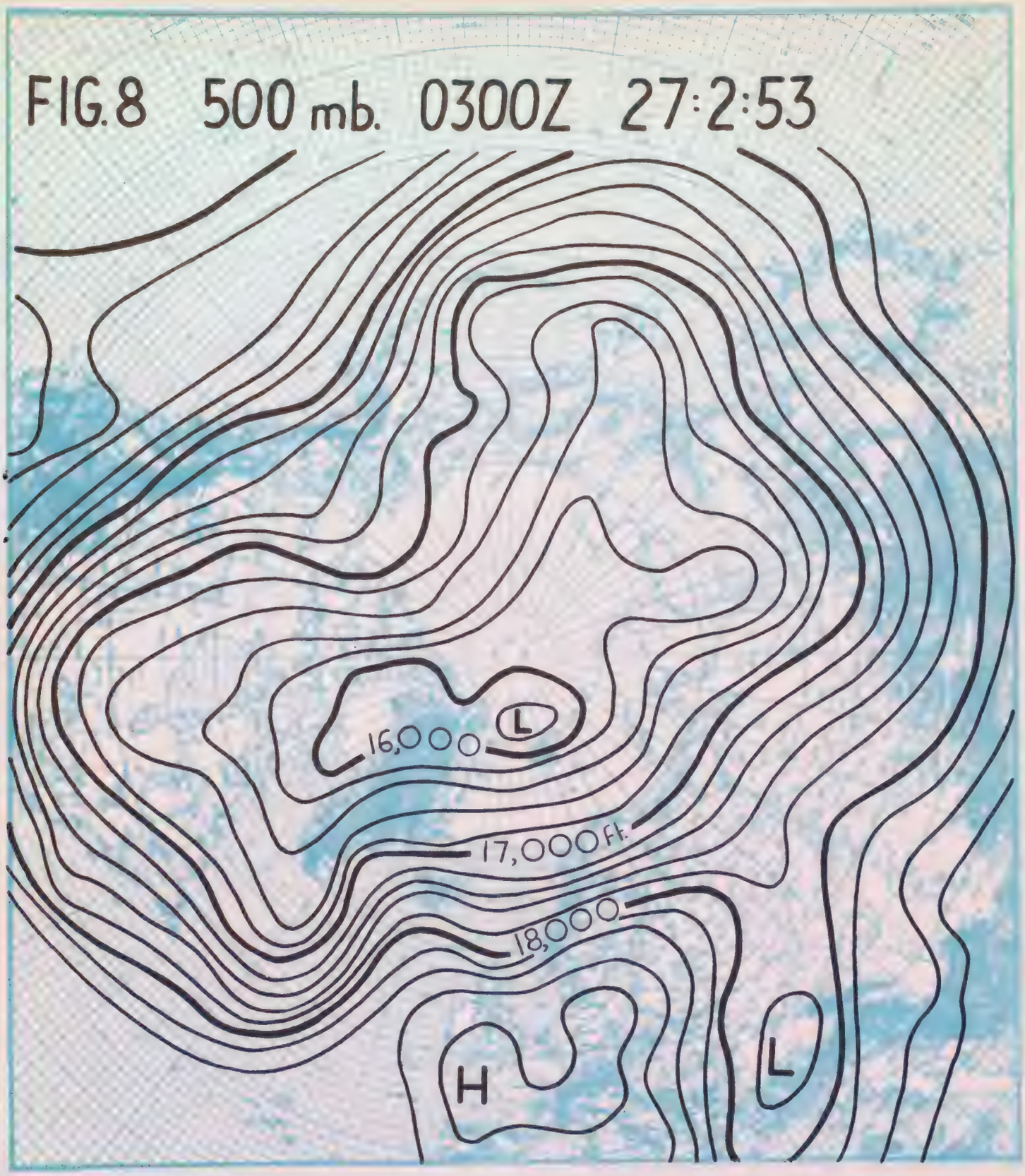


FIG. 8 500 mb. 0300Z 27:2:53
16,000
L
17,000 Ft.
18,000
H
L

FIG. 9 500mb. 0300Z 27:2:53

SPHERICAL HARMONIC REPRESENTATION $n,m,\leq 12$

FIG. 10 500mb. 0300Z 27:2:53
SPHERICAL HARMONIC REPRESENTATION $n,m,\leq 8$

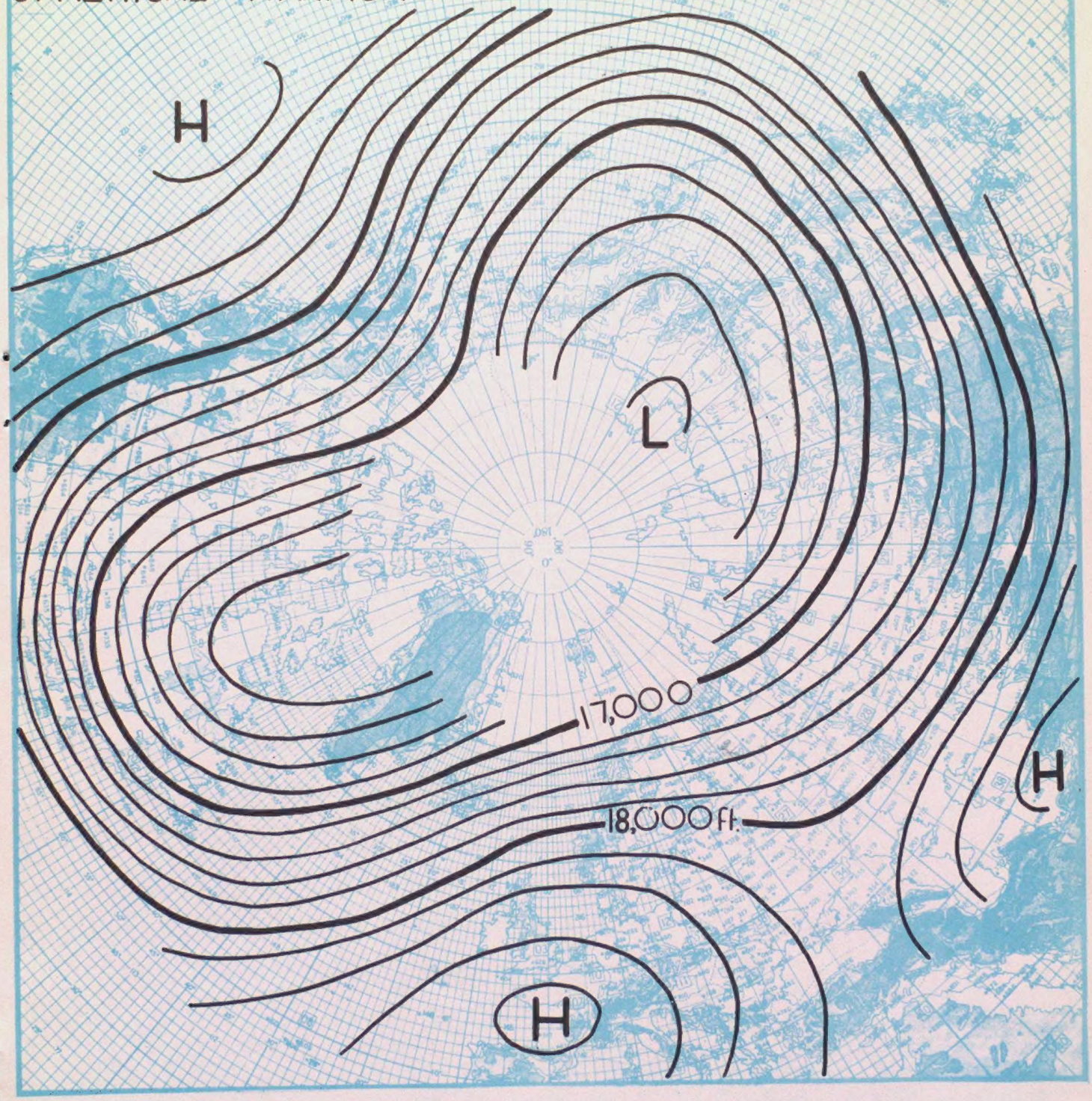


FIG. 11 500mb. 0300Z 29:6:53

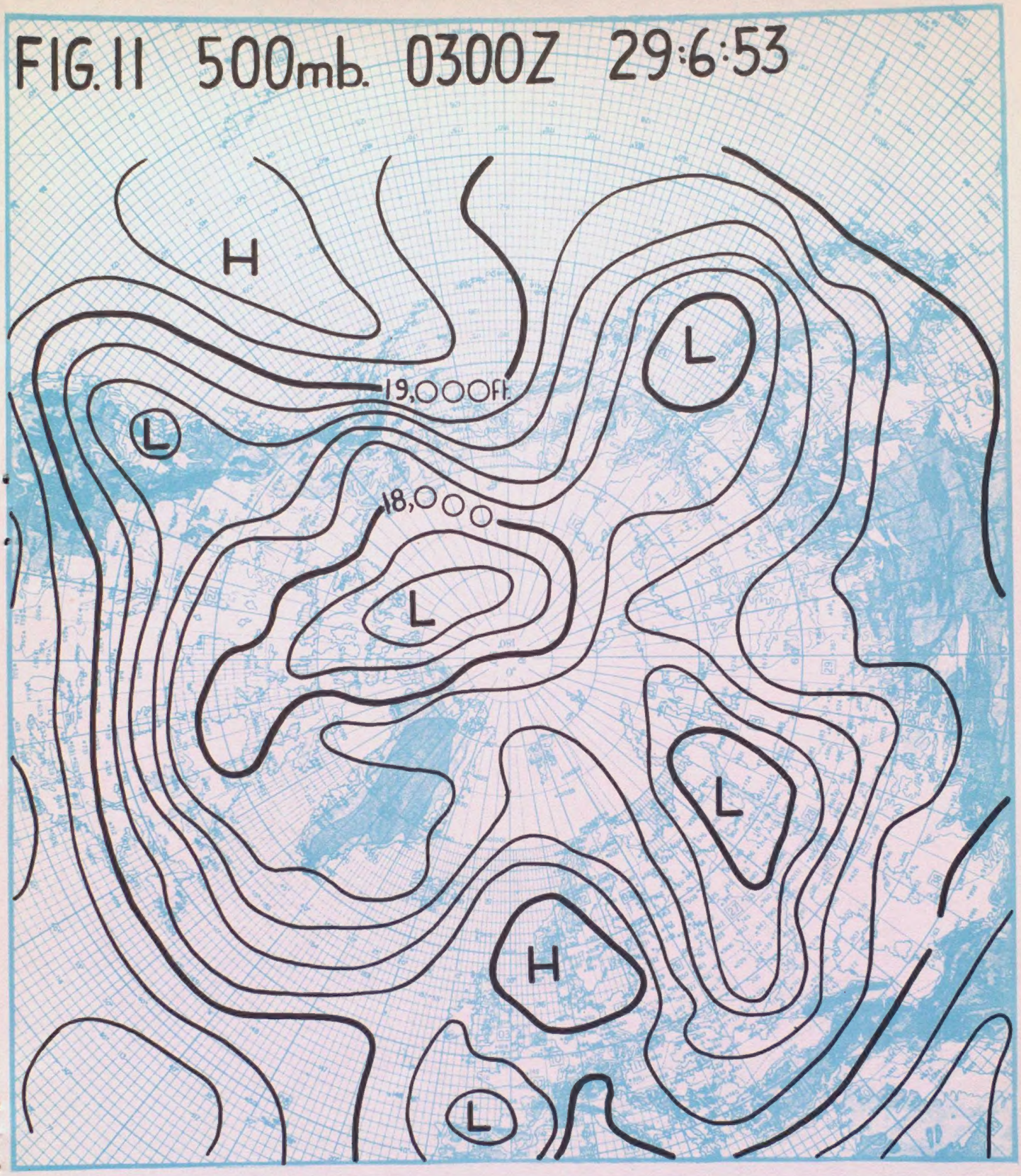


FIG.12 500mb. 0300Z 29:6:53

SPHERICAL HARMONIC REPRESENTATION $n,m,\leq 12$

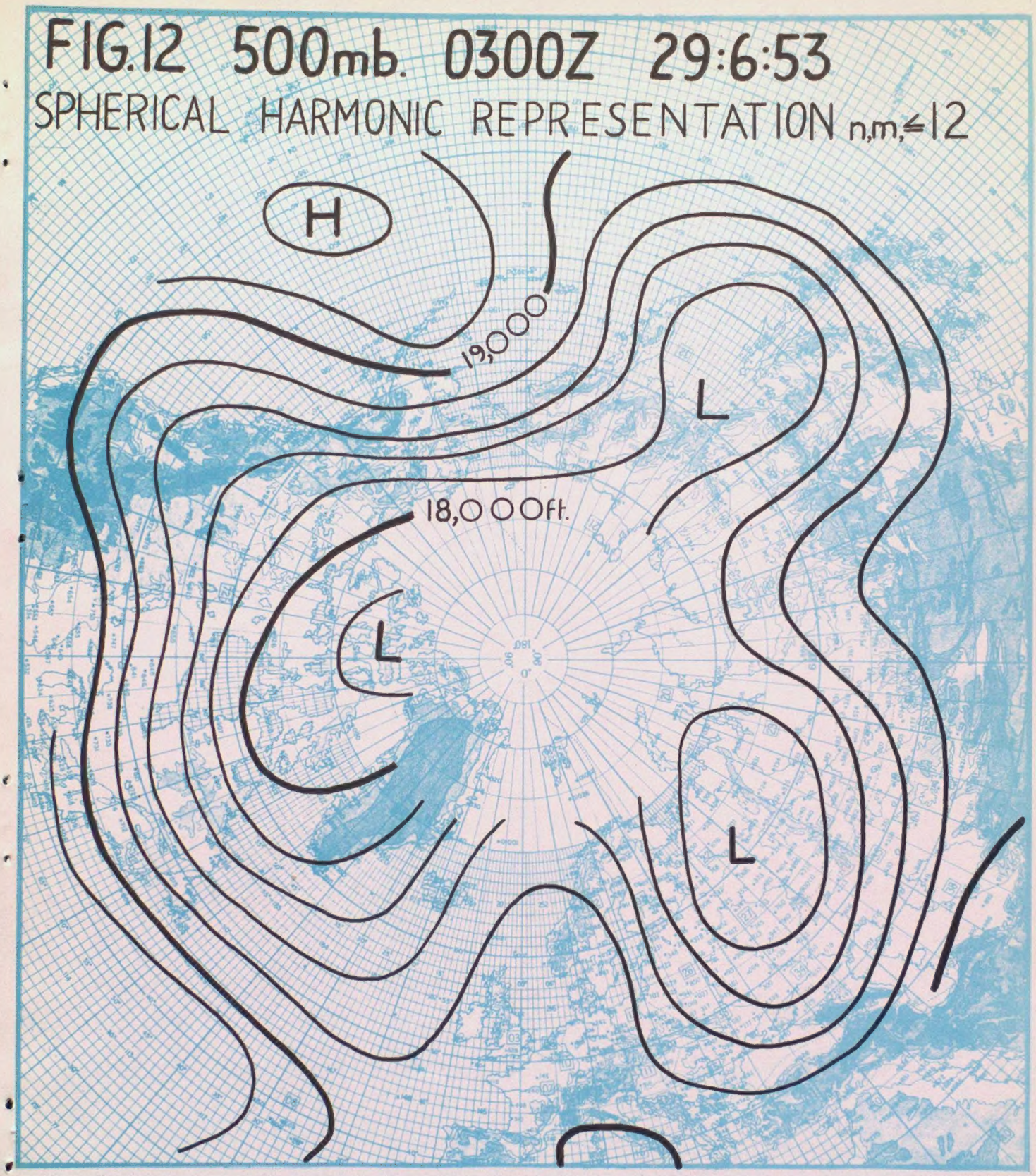
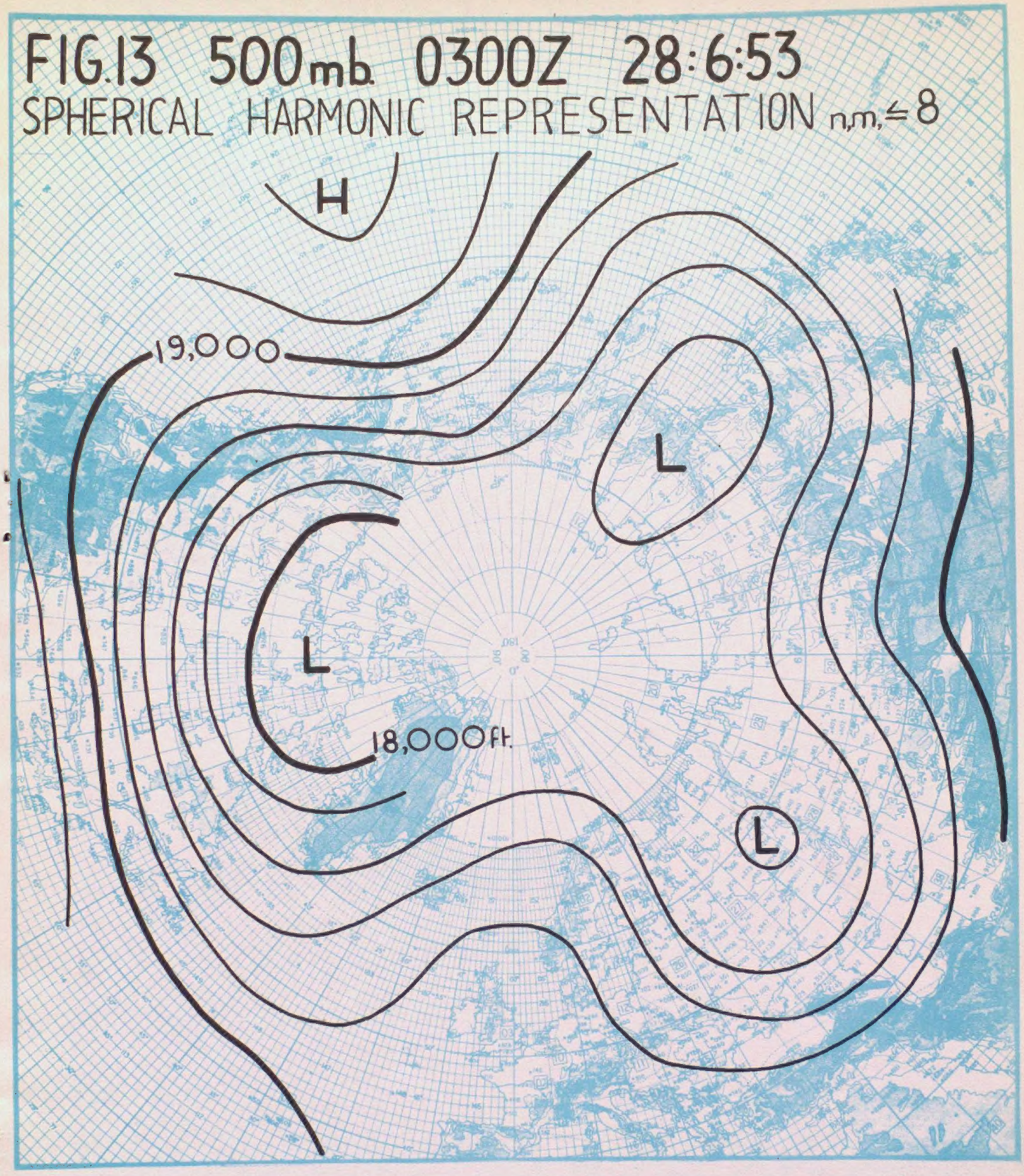


FIG.13 500mb. 0300Z 28:6:53

SPHERICAL HARMONIC REPRESENTATION $n,m, \leq 8$